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ON THE PROBLEM OF THE COUPLED CAVITY CHAIN CHARACTERISTIC CALCULATIONS

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Abstract

This paper presents the results of studies in elaboration of a mathematical model of the cavity-chain slow wave structures. Considered is the problem of coupling of an infinitely long cylindrical cavity chain coupled through centerholes in the dividing walls of finite thickness without the assumption about smallness of any parameters. On the basis of a rigorous electrodynamic approach it is shown that the cavity chain can be described by a set of equations that describe the coupling of an infinite number of similar-type modes of short-circuit cavities. At the same time the coupling coefficients are determined by solving the basic sets of linear algebraic equations which describe the coupling the tangential electric fields on the right and left cylindrical hole cross-sections of the disks. The results are given of numerical simulations of the dependence of electrodynamic characteristics on the number of connections between the modes taken into account. Based on these calculations for different cavity dimensions, the number of coupling connections that give the sufficient accuracy of the description was determined.

1 Introduction

The chain of coupled cavities are widely used in the RF-engineering. Slow wave structures on their base are most common in the accelerator technology, finding various applications as well in RF-devices designed to generate and amplify electromagnetic waves (see, for example [1]-[3]). Since resonant properties of each cavity can be described by equations that resemble in their outward appearance the classic equations of the resonant circuits, then, a coupled cavity chain should be described similarly. Such an approach to the study of properties of coupled cavity chain (a method of equivalent circuits) is very useful to model rf cavities. Its advantage over purely electrodynamic methods is in its explicitness and a relative simplicity of the mathematical analysis which is of supreme importance for the stage of primary electrodynamic properties study and conceptual design of the system. It is especially manifest in the development of complex structures: a chain of cavities coupled through slots [4], a biperodic or compensated structure [5, 6], a detuned structure [7, 8] and others. However, justification of such models must be made on the base of the electrodynamic approach which simultaneously gives their accuracy.

The main question in utilization of the circuit model for description of a coupled cavity chain is the possibility of truncation the number of circuits under consideration and their connections, since a precise account of all these factors would do more than simply eliminate the advantages of this approach: it would make this problem mathematically unresolvable.

By doing such truncation we have to take into account the following circumstance. Since commonly the analysis of characteristics has to be made within a confined frequency range, then the total initial set of circuits can be broken down into two classes: the resonant one, representing modes with eigen-frequencies which values are close to the frequency range under the consideration, and the non-resonant one. If the couplings of the resonant circuits form the main frequency properties of the system, then the presence of the non-resonant circuits form the properties of such couplings. Although the amplitude

of each non-resonant mode is small , their total effect on resonant circuit couplings is considerable.

Based on the rigorous electrodynamic approach, in two cavity coupling problem we could separate the above circuit types and also bring down the study of influence of the non-resonant modes to the field coupling on the boundaries dividing the cavities ([9]-[11]). This approach allowed to preserve the explicitness of the model as a system of coupled resonant circuits and calculate accurately the necessary coupling coefficients.

In this work this method is used to describe an infinitely long chain of cylindrical cavities coupled through central holes in sidewalls. The focus of attention is paid to the calculation of the value of coupling of cavities which have no immediate contact. The number of couplings to be taken into account determines not only the slow-wave structure properties, but also a possibility of their matching and tuning. The latter is of added importance for development of inhomogeneous structures. As distinct from the previous papers ([3], [5, 6], [12]), we have managed to elaborate on a model, allowing consecutively to take into consideration any number of couplings.

2   Problem Definition

Let us consider an infinite chain of similar ideally conducting co-axial cylindrical cavities (disk-loaded wavequide) coupled through cylindrical holes with the radius a in the dividing walls of the thickness t. The radii and lengths of the cavities we denote by b d. The disks of the i-th cavity we denote by the indexes i $i-1$. In order to construct a mathematical system model under consideration we will use a method of partial cross-over regions [13]. As a main set of regions, we take the cylindrical cavity volumes; as an auxiliary one — the cylinders that are co-axial with coupling holes and of the same radii[2] $b' = a$. Each of these cylinders projects into the volumes of the adjacent cavities over the length d_*. The lengths of these cavities we denote by d' ($d' = 2d_* + t$). In each region of the two sets we expand the electromagnetic field with the short-circuit resonant cavity modes[3]. Below, we will consider only axially-symmetric -fields, that is why the irrotational mode amplitudes in the appropriate expansions will be zero. In this case for the main regions (i is the cavity number) the expansions will be as follows:

$$\vec{E}^i = \sum_{n,s} e_{n,s}^{(i)} \vec{\mathcal{E}}_{n,s}^{(i)} \tag{1}$$

$$\vec{H}^i = \sum_{n,s} h_{n,s}^{(i)} \vec{\mathcal{H}}_{n,s}^{(i)} \tag{2}$$

where

$$\mathcal{E}_{n,s,z}^{(i)} = \frac{\lambda_s^2 c}{b^2 \omega_{n,s} N_{n,s}} \cos[k_n(z - D \times i)] J_0\left(\lambda_s r / b\right), \tag{3}$$

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[2]All auxiliary region-related values will marked by the prime, for instance, b', $\omega_{n',s'}'$

[3]If we employ such expansion we must observe the fact that this expansion is complete for total electric field within the cavity, but it is incapable of representing tangential electric field on the hole, so that is the reason why the condition $d_* < d$ should be imposed on d_*, at least, during derivation of the set of equations (see below)

$$\mathcal{E}_{n,s,\mathrm{r}}^{(i)} = \frac{\lambda_s k_n c}{b\omega_{n,s}N_{n,s}}\sin[k_n(z - D \times i)]J_1\left(\lambda_s r/b\right), \tag{4}$$

$$\mathcal{H}_{n,s,\phi}^{(i)} = -i\frac{\lambda_s}{bN_{n,s}}\cos[k_n(z - D \times i)]J_1\left(\lambda_s r/b\right), \tag{5}$$

$$s = 1, 2, 3\ldots\infty; \ n = 0, 1, 2\ldots\infty; \ J_0\left(\lambda_s\right) = 0;$$

$$\omega_{n,s} = c\sqrt{k_n^2 + \lambda_s^2/b^2}; \ k_n = \pi n/d; \ N_{n,s} = \sqrt{\pi\epsilon_n d\lambda_s^2 J_1^2(\lambda_s)/2};$$

$$\epsilon_n = \left\{ \begin{array}{ll} 2, & n = 0, \\ 1, & n \neq 0, \end{array} \right.$$

The mode set (3-5) satisfies the following orthonormality conditions :

$$\int_v \vec{\mathcal{E}}_{n,s}^{(i)}\vec{\mathcal{E}}_{n',s'}^{(i)*}dV = \int_v \vec{\mathcal{H}}_{n,s}^{(i)}\vec{\mathcal{H}}_{n',s'}^{(i)*}dV = \delta_{n,n'}\delta_{s,s'}. \tag{6}$$

For the auxiliary regions the field expansion takes on the form similar to (1,2), with the eigen-functions being derived from the formulae (3–5) by the way of substitution $b \to a, d \to d'$.

The coefficients in the expansion (1) are determined by the tangential components of electric field on the boundaries of regions of choice

$$\left(\omega_{n,s}^{(i)2} - \omega^2\right)e_{n,s}^{(i)} = -ic\omega_{n,s}^{(i)}\int_S \left[\vec{E}\vec{\mathcal{H}}_{n,s}^{*(i)}\right]d\vec{s}. \tag{7}$$

Since the electric field tangential component on a metallic surface is zero, then, in Eq.(7) the integration surfaces for the main regions will be circles located on the opposite cavity walls, while for the auxiliary ones — two cylindrical surfaces and two circles, over which these regions are in contact with the former two regions. Remembering this, we derive from (7) the following:

$$\left(\omega_{k,l}^2 - \omega^2\right)e_{k,l}^{(i)} = \sum_{n',s'}\left(e_{n',s'}^{\prime\,(i)}L_{n',s',k,l}^{(1)} + e_{n',s'}^{\prime\,(i-1)}L_{n',s',k,l}^{(2)}\right), \tag{8}$$

$$e_{n',s'}^{\prime\,(i)} = \sum_{n,s}\left(e_{n,s}^{(i)}T_{n,s,n',s'}^{(1)} + e_{n,s}^{(i+1)}T_{n,s,n',s'}^{(2)}\right), \tag{9}$$

where

$$L_{n',s',k,l}^{(1)} = -ic\omega_{k,l}2\pi\int_0^a rdr\left(\mathcal{E}_{n',s',\mathrm{r}}^{\prime\,(i)}\mathcal{H}_{k,l,\phi}^{(i)\,*}\right)_{z=d+D\times i},$$

$$L_{n',s',k,l}^{(2)} = ic\omega_{k,l}2\pi\int_0^a rdr\left(\mathcal{E}_{n',s',\mathrm{r}}^{\prime\,(i-1)}\mathcal{H}_{k,l,\phi}^{(i)\,*}\right)_{z=D\times i},$$

$$T_{n,s,n',s'}^{(1)} = \frac{2\pi ic\omega_{n',s'}'}{\omega_{n',s'}'^2 - \omega^2}\times$$

$$\times\left[-a\int_{d-d_*+D\times i}^{d+D\times i}dz\left(\mathcal{E}_{n,s,z}^{(i)}\mathcal{H}_{n',s',\phi}^{\prime\,(i)\,*}\right)_{r=a} - \int_0^a rdr\left(\mathcal{E}_{n,s,\mathrm{r}}^{(i)}\mathcal{H}_{n',s',\phi}^{\prime\,(i)\,*}\right)_{z=d-d_*+D\times i}\right],$$

$$T_{n,s,n',s'}^{(2)} = \frac{2\pi ic\omega_{n',s'}'}{\omega_{n',s'}'^2 - \omega^2}\times$$

$$\times \left[-a \int_{D+D\times i}^{D+d_*+D\times i} dz \left(\mathcal{E}_{n,s,z}^{(i+1)} \mathcal{H}_{n',s',\phi}^{\prime\,(i)\,*} \right)_{r=a} - \int_0^a r dr \left(\mathcal{E}_{n,s,r}^{(i+1)} \mathcal{H}_{n',s',\phi}^{\prime\,(i)\,*} \right)_{z=D+d_*+D\times i} \right].$$

By substituting (9) into (8), and also by introducing new variables instead of $e_{k,\ell}^{(i)}$ for convenience

$$a_{k,l}^{(i)} = e_{k,l}^{(i)} \frac{\lambda_l J_0\left(\lambda_l a/b\right)}{\omega_{k,l}\sqrt{\epsilon_k}J_1(\lambda_l)},$$

we derive a set of equations for mode amplitudes only in the main regions:

$$\epsilon_k Z_{k,l} a_{k,l}^{(i)} = \sum_{n,s} a_{n,s}^{(i)} \left(V_{n,s,k,l}^{(1,1)} + V_{n,s,k,l}^{(2,2)} \right) +$$

$$+ \sum_{n,s} \left(a_{n,s}^{(i+1)} V_{n,s,k,l}^{(1,2)} + a_{n,s}^{(i-1)} V_{n,s,k,l}^{(2,1)} \right) \tag{10}$$

where $Z_{k,l} = \omega_{k,l}^2 - \omega^2$,

$$V_{n,s,k,l}^{(i,j)} = (-1)^{1+i\times(1+k)+j\times(1+n)} \alpha \gamma_l \times$$

$$\times \sum_{s'} \sigma_{s',l} \Delta_{s',n} \left[f_{s'}^{(i,j)} - \beta Z_{n,s} \sigma_{s',s} B_n F_{s'}^{(i,j)} \right],$$

$$\alpha = 4a^3 c^2 / \left(b^4 d \right), \gamma_l = \lambda_l^2 J_0^2 \left(\lambda_l a/b \right) / J_1(\lambda_l),$$

$$\sigma_{s,l} = \left(\lambda_s^2 - a^2 \lambda_l^2/b^2 \right)^{-1}, \Delta_{s,n} = \left[\lambda_s^2 - \Omega^2 + (\pi a n/d)^2 \right]^{-1},$$

$$\beta = 2a^3 / \left(c^2 d \right), B_n = \pi n \sin(\pi n d_*/d),$$

$$F_s^{(i,j)} = \frac{1}{\sinh(q_s)} \begin{cases} \sinh[q_s \left(1 - d_*/d' \right)], & i = j, \\ \sinh(q_s d_*/d'), & i \neq j, \end{cases}$$

$$f_s^{(i,j)} = \frac{\mu_s}{\sinh(q_s)} \begin{cases} \cosh(q_s) - \cosh[q_s \left(1 - 2d_*/d' \right)], & i = j, \\ \cosh(q_s 2d_*/d') - 1, & i \neq j, \end{cases}$$

$$q_s = \mu_s d'/a, \ \mu_s = \sqrt{\lambda_s^2 - \Omega^2}, \ \Omega = \omega a/c.$$

The homogeneous set of Eqs.(10) describes the coupling of infinite set of resonant circuits which are the short-circuit resonant cavity modes, and, in principle, it can be used to calculate the necessary electrodynamic characteristics of a cavity chain. However, the set (10) has a few drawbacks that make it difficult to carry out both analytical investigations and numerical calculations. Firstly, the structure of this set of equations does not yield a possibility to obtain analytical results, in particular, it is difficult to take account the non-resonant modes. Secondly, this set is two-dimensional, and it is necessary to have great calculative resources to solve it. Our studies show that the set of Eqs.(10) can be reduced to such a form that would permit analytical studies to be made and simplify the numerical calculations. Below we present the results of our studies.

3 Derivation of the main set of equations

We shall seek the amplitudes $a_{k,l}^{(i)}$, with the exception of the fundamental modes $((k,l) \neq (0,1))$, as:

$$\epsilon_k Z_{k,l} a_{k,l}^{(i)} = \sum_m a_{0,1}^{(m)} x_{k,l}^{(i,m)}. \tag{11}$$

Introducing instead of one sequence of unknowns $\left\{ a_{k,l}^{(i)} \right\}$, $(i = -\infty, \ +\infty)$ an infinite number of new ones $\left\{ x_{k,l}^{(i,m)} \right\}$, $(i = -\infty, +\infty, \ m = -\infty, +\infty)$, we can impose $(\infty - 1)$ additional conditions on the new sequences. We shall consider that $\left\{ x_{k,l}^{(i,m)} \right\}$ satisfy the equations

$$x_{k,l}^{(i,m)} - \sideset{}{'}\sum_{n,s} \frac{1}{\epsilon_n Z_{n,s}} \left[x_{n,s}^{(i,m)} \left(V_{n,s,k,l}^{(1,1)} + V_{n,s,k,l}^{(2,2)} \right) + \right.$$

$$\left. + x_{n,s}^{(i+1,m)} V_{n,s,k,l}^{(1,2)} + x_{n,s}^{(i-1,m)} V_{n,s,k,l}^{(2,1)} \right] =$$

$$= \left(V_{0,1,k,l}^{(1,1)} + V_{0,1,k,l}^{(2,2)} \right) \delta_{m,i} + V_{0,1,k,l}^{(1,2)} \delta_{m,i+1} + V_{0,1,k,l}^{(2,1)} \delta_{m,i-1}, \tag{12}$$

where $(k,l) \neq (0,1)$. In Eqs.(12) and elsewhere below the prime in sums indicate that $(n,s) \neq (0,1)$. In so doing, from (10) it follows that the amplitudes of the fundamental modes $((k,l) = (0,1))$ should satisfy the equations

$$2Z_{0,1} a_{0,1}^{(i)} = a_{0,1}^{(i)} \left(V_{0,1,0,1}^{(1,1)} + V_{0,1,0,1}^{(2,2)} \right) + a_{0,1}^{(i+1)} V_{0,1,0,1}^{(1,2)} + a_{0,1}^{(i-1)} V_{0,1,0,1}^{(2,1)} +$$

$$+ \sum_m a_{0,1}^{(m)} \sideset{}{'}\sum_{n,s} \frac{1}{\epsilon_n Z_{n,s}} \left[x_{n,s}^{(i,m)} \left(V_{n,s,0,1}^{(1,1)} + V_{n,s,0,1}^{(2,2)} \right) + \right.$$

$$\left. + x_{n,s}^{(i+1,m)} V_{n,s,0,1}^{(1,2)} + x_{n,s}^{(i-1,m)} V_{n,s,0,1}^{(2,1)} \right]. \tag{13}$$

Let's denote

$$v_{+,s}^{(i,m)} = f_s^{(1,1)} p_{+,s}^{(i+1,m)} - f_s^{(1,2)} p_{-,s}^{(i,m)} - F_s^{(1,1)} q_{+,s}^{(i+1,m)} + F_s^{(1,2)} q_{-,s}^{(i,m)}, \tag{14}$$

$$v_{-,s}^{(i,m)} = f_s^{(1,1)} p_{-,s}^{(i,m)} - f_s^{(1,2)} p_{+,s}^{(i+1,m)} - F_s^{(1,1)} q_{-,s}^{(i,m)} + F_s^{(1,2)} q_{+,s}^{(i+1,m)}, \tag{15}$$

where

$$\sideset{}{'}\sum_{n,s} \frac{x_{n,s}^{(i,m)}}{\epsilon_n Z_{n,s}} \Delta_{s',n} = q_{+,s'}^{(i,m)} - \delta_{i,m} \Delta_{s',0}$$

$$\sideset{}{'}\sum_{n,s} (-1)^n \frac{x_{n,s}^{(i,m)}}{\epsilon_n Z_{n,s}} \Delta_{s',n} = q_{-,s'}^{(i,m)} - \delta_{i,m} \Delta_{s',0}$$

$$\sideset{}{'}\sum_{n,s} \frac{x_{n,s}^{(i,m)}}{\epsilon_n} \Delta_{s',n} \sigma_{s',s} B_n = p_{+,s'}^{(i,m)}$$

$$\sideset{}{'}\sum_{n,s} (-1)^n \frac{x_{n,s}^{(i,m)}}{\epsilon_n} \Delta_{s',n} \sigma_{s',s} B_n = p_{-,s'}^{(i,m)}$$

Then, the set of equations (13) can be written down as:

$$2Z_{0,1}a_{0,1}^{(i)} = -\alpha\gamma_1\sum_m a_{0,1}^{(m)}\sum_{s'}\sigma_{s',1}\left[v_{+,s'}^{(i-1,m)} + v_{-,s'}^{(i,m)}\right],\tag{16}$$

As follows from (12), the coefficients $v_{+,s}^{(i,m)}$ and $v_{-,s}^{(i,m)}$ satisfy the equations

$$v_{+,n}^{(i,m)} + \sum_s\left[v_{+,s}^{(i,m)}G_{n,s}^{(1,1)} - v_{-,s}^{(i,m)}G_{n,s}^{(1,2)} + v_{-,s}^{(i+1,m)}G_{n,s}^{(2,1)} - v_{+,s}^{(i-1,m)}G_{n,s}^{(2,2)}\right] =$$

$$= f_n^{(1,1)}\delta_{i+1,m}\Delta_{n,0} - f_n^{(1,2)}\delta_{i,m}\Delta_{n,0},\tag{17}$$

$$v_{-,n}^{(i,m)} + \sum_s\left[v_{-,s}^{(i,m)}G_{n,s}^{(1,1)} - v_{+,s}^{(i,m)}G_{n,s}^{(1,2)} + v_{+,s}^{(i-1,m)}G_{n,s}^{(2,1)} - v_{-,s}^{(i+1,m)}G_{n,s}^{(2,2)}\right] =$$

$$= f_n^{(1,1)}\delta_{i,m}\Delta_{n,0} - f_n^{(1,2)}\delta_{i+1,m}\Delta_{n,0},\tag{18}$$

where

$$G_{n,s}^{(i,j)} = f_n^{(1,j)}\mathcal{T}_{n,s}^{(i)} - F_n^{(1,j)}\mathcal{L}_{n,s}^{(i)},\tag{19}$$

$$\mathcal{L}_{n,s}^{(1)} = \alpha\beta{\sum_{k,l}}'\gamma_l\sigma_{s,l}\sigma_{n,l}\Delta_{n,k}B_k/\epsilon_k = \delta_{n,s}\frac{\sinh[(d-d_*)\mu_n/a]}{\sinh(d\mu_n/a)},\tag{20}$$

$$\mathcal{L}_{n,s}^{(2)} = \alpha\beta{\sum_{k,l}}'(-1)^k\gamma_l\sigma_{s,l}\sigma_{n,l}\Delta_{n,k}B_k/\epsilon_k == -\delta_{n,s}\frac{\sinh(d_*\mu_n/a)}{\sinh(d\mu_n/a)},\tag{21}$$

$$\mathcal{T}_{n,s}^{(1)} = \alpha{\sum_{k,l}}'\gamma_l\sigma_{s,l}\Delta_{n,k}/\left(\epsilon_k Z_{k,l}\right),$$

$$\mathcal{T}_{n,s}^{(2)} = \alpha{\sum_{k,l}}'(-1)^k\gamma_l\sigma_{s,l}\Delta_{n,k}/\left(\epsilon_k Z_{k,l}\right),$$

$$\mathcal{T}_{n,s}^{(i)} = \pi\frac{a}{b}\sum_{l=1}^{\infty}\frac{\theta_l^3 J_0^2(\theta_l)E_l^{(i)}\left(a/d,\nu_l\right)}{\chi_l\left(\lambda_n^2 - \theta_l^2\right)\left(\lambda_s^2 - \theta_l^2\right)} - \frac{1}{2}\delta_{n,s}E_2^{(i)}\left(a/d,\mu_n\right) +$$

$$+ \frac{\pi a^2}{\mu_n^2 bd}\frac{\theta_1^3 J_0^2(\theta_1)}{\left(\lambda_n^2 - \theta_1^2\right)\left(\lambda_s^2 - \theta_1^2\right)},\tag{22}$$

$$E_l^{(1)}(x,y) = \begin{cases} \coth(y/x)/y - x/y^2, & l = 1, \\ \coth(y/x)/y, & l \neq 1, \end{cases}$$

$$E_l^{(2)}(x,y) = \begin{cases} \sinh^{-1}(y/x)/y - x/y^2, & l = 1, \\ \sinh^{-1}(y/x)/y, & l \neq 1, \end{cases}$$

$$\theta_l = a\lambda_l/b,\ \chi_l = \pi\lambda_l J_1^2(\lambda_l)/2,\ \nu_l = \sqrt{\theta_l^2 - \Omega^2},\ \mu_l = \sqrt{\lambda_l^2 - \Omega^2}.$$

By comparing Eq.(7) and Eq.(16) one can deduce that the electric field tangential components in the circular regions, through which i-th cavity is connected with others elements of the system under consideration , are only determined via the fundamental mode (E_{010}) amplitudes of all cavities. The coefficients $v_{+,s}^{(i-1,m)}$, determining the fields on the left side wall of the i-th cavity (the right hole cross-section of the $i-1$-th disk), and $v_{-,s}^{(i,m)}$, determining the fields on the right side wall of the i-th cavity (the left hole

cross-section of the i-th disk), are proportional to the expansion coefficients of tangential electric field with the complete set of functions $\{J_1\left(\lambda_s r/a\right)\}$.

Thus, the problem of coupled cavities has been rigorously reduced to the problem of the coupling of electric fields (see, Eqs.(17,18)), which are determined in circular regions $r \le a$, $z = D \times i$, $z = d + D \times i$, $i = -\infty, +\infty$. As Eqs.(17,18) indicate, immediately coupled are only four fields: on the right and left cross-sections of the i-th disk, on the left cross-section of the $(i+1)$-th disk and on the right cross-section of the $(i-1)$-th disk. This is a consequence of the fact that the i-th cavity in the chain is in immediate contact with only two adjacent cavities ($i+1$ and $i-1$). It can be deduced from Eqs.(17,18) that the coefficients $v_{+,s}^{(i,m)}$, $v_{-,s}^{(i,m)}$ obey the following symmetry conditions:

$$v_{\pm,s}^{(i,m)} = v_{\pm,s}^{(i+k,m+k)}, \tag{23}$$

$$v_{+,s}^{(m-1-k,m)} = v_{-,s}^{(m+k,m)}, \tag{24}$$

where k is any integer. The correlation (23) is a reflection of translational symmetry of the system under consideration (field induced by the m-th cavity on the i-th disk does not change during the simultaneous shift of the cavity and disk numbers), while the correlation (24) is a reflection of considered mode symmetry (field induced by the m-th cavity on the left cross-section of the $(m + k)$-th disk is equal to the field on the righ cross-section of the $(m - 1 - k)$-th disk). Taking into account these correlations and introducing a new count of disks to the right from the i-th cavity ($k = 1, 2, 3 \ldots \infty$), we obtain

$$\left(\omega_{0,1}^2 - \omega^2\right) a_{0,1}^{(i)} = -\omega_{0,1}^2 \frac{2}{3\pi} \frac{a^3}{J_1^2(\lambda_1)b^2 d} \times$$

$$\times \left[2a_{0,1}^{(i)}\Lambda_{-,1} + \sum_{k=1}^{\infty} \left(a_{0,1}^{(i+k)} + a_{0,1}^{(i-k)}\right)\left(\Lambda_{-,k+1} - \Lambda_{+,k}\right)\right], \tag{25}$$

where the normalized coupling coefficients are determined by the formulae:

$$\Lambda_{\pm,k} = J_0^2\left(\lambda_1 a/b\right)\sum_{s'} w_{\pm,s'}^{(k)}/\left[\lambda_{s'}^2 - (\lambda_1 a/b)^2\right], \tag{26}$$

while field coupling over the semi-infinite disk number are described by ($w_{\pm,s}^{(k)} = \mp v_{\pm,s}^{(i+k-1,i)} = \mp v_{\pm,s}^{(k-1,0)}$)

$$w_{+,n}^{(1)} + \sum_{s}\left[w_{+,s}^{(1)}G_{n,s}^{(1,1)} + w_{-,s}^{(1)}G_{n,s}^{(1,2)} + w_{-,s}^{(1)}G_{n,s}^{(2,2)}\right] =$$

$$= \sum_{s} w_{-,s}^{(2)}G_{n,s}^{(2,1)} + f_n^{(1,2)}3\pi/\left[\lambda_n^2 - (\lambda_1 a/b)^2\right], \tag{27}$$

$$w_{-,n}^{(1)} + \sum_{s}\left[w_{-,s}^{(1)}G_{n,s}^{(1,1)} + w_{+,s}^{(1)}G_{n,s}^{(1,2)} + w_{-,s}^{(1)}G_{n,s}^{(2,1)}\right] =$$

$$= \sum_{s} w_{-,s}^{(2)}G_{n,s}^{(2,2)} + f_n^{(1,1)}3\pi/\left[\lambda_n^2 - (\lambda_1 a/b)^2\right], \tag{28}$$

$$w_{+,n}^{(k)} + \sum_{s}\left[w_{+,s}^{(k)}G_{n,s}^{(1,1)} + w_{-,s}^{(k)}G_{n,s}^{(1,2)}\right] = \sum_{s}\left[w_{-,s}^{(k+1)}G_{n,s}^{(2,1)} + w_{+,s}^{(k-1)}G_{n,s}^{(2,2)}\right], \tag{29}$$

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$$w_{-,n}^{(k)} + \sum_s \left[w_{-,s}^{(k)} G_{n,s}^{(1,1)} + w_{+,s}^{(k)} G_{n,s}^{(1,2)} \right] = \sum_s \left[w_{+,s}^{(k-1)} G_{n,s}^{(2,1)} + w_{-,s}^{(k+1)} G_{n,s}^{(2,2)} \right]. \qquad (30)$$

In Eqs.(29,30) $k = 2, 3 \ldots \infty$.

The closed set of equations (25-30) describes rigorously the electrodynamic system under consideration. Eqs.(25) describe the coupling of the infinite chain of the resonant circuits, with the coupling coefficients $\Lambda_{\pm,k}$ being frequency functions.

4   Results of the Analysis and Numerical Simulation

As mentioned above, the expansion into the Fourier series with the short-circuit resonant cavity modes is complete for total electric field within the cavity, but it is incapable of representing tangential electric field on the hole, that is why d_* has to satisfy the condition $d_* < d$. Yet, as analysis and computer simulation indicate, in the final formulae we may assume that $d_* = d$. It is connected with the fact, that the value $\mathcal{L}_{n,s}^{(2)}$ as function d_* has a discontinuity, being equal to zero at $d_* = d$ and to finite value at $d_* \neq d$. The expression, determining $\mathcal{L}_{n,s}^{(2)}$ (see(22)), at $d_* < d$ has a finite limit in the case $d_* \to d$: $\lim_{d_* \to d} \mathcal{L}_{n,s}^{(2)} = -\delta_{n,s}$, using which gives the correct results even at $d_* = d$. The numerical simulations indicate that at $10^{-7} < d_* \leq d$ the calculation results depend on the value d_* only in the seventh digit. In the case $d_* = d$ — $\mathcal{L}_{n,s}^{(1)} = 0$ and, if we put $G_{n,s}^{(2,i)} = 0$ $(i = 1, 2)$ the Eqs.(27,28) will describe the field coupling on the disk dividing two equal cylindrical cavities (see [8]-[10]). Thus, the coupling of fields on various disks are described in Eqs.(27,30) by the terms which contain factors $G_{n,s}^{(2,i)}$ $(i = 1, 2)$. This is confirmed as well by the fact that at $a \to 0$ and $t = 0$ $G_{n,s}^{(2,i)} \to 0$ $(i = 1, 2)$, while $G_{n,s}^{(1,i)}$ $(i = 1, 2)$ tend to constant values, independent of a:

$$G_{n,s}^{(1,i)} = \lambda_n \int_0^\infty d\theta \frac{\theta^2 J_0^2(\theta)}{\left(\lambda_n^2 - \theta^2\right)\left(\lambda_s^2 - \theta^2\right)} - \frac{1}{2}\delta_{n,s}, \ (i = 1, 2). \qquad (31)$$

In this case Eqs.(27,30) have the following solution

$$w_{+,n}^{(1)} = w_{-,n}^{(1)} = 6\left(\sin(\lambda_n) - \lambda_n \cos(\lambda_n)\right)/\lambda_n^2,$$

$$w_{+,n}^{(k)} = w_{-,n}^{(k)} = 0, \ k = 2, 3 \ldots \infty,$$

at which $\Lambda_{-,1} = \Lambda_{+,1} = 1$ and all the remaining values of the normalized coupling coefficients are equal to zero. At these conditions the set (25) coincides with the equations describing an infinite cavity chain obtained in the quasi-static approximations [14]-[16].

The structure of Eqs.(27,28), determining fields on the first disk is different from that (29,30) which determine fields on the subsequent disks not only in the presence of the "driving" force, but also in the nature of coupling. Fields on the first disk couple only with fields on the second disk while fields on the disk, beginning from the second one, couple with fields of two adjacent disks. This fact is associated with transformation of infinite set of equations into semi-infinite one in accordance with the field symmetry. Indeed, the coupling of the first disk fields with the fields on the right cross-section of (-1)-th disk in accordance with the field symmetry has transformed into a "self-coupling", described by the terms $w_{-,s}^{(1)} G_{n,s}^{(2,2)}$ in (27) and $w_{-,s}^{(1)} G_{n,s}^{(2,1)}$ in (28). This taken into account, from the set

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(25,30)) at $t \to \infty$ one can obtain the equations describing the characteristics of a single cavity with two drift tubes. In this case, owing to the field coupling, the frequency shift with two tube will not be equal to the doubled frequency shift with one tube.

Eqs.(25) have the solution of the kind $a_{0,1}^{(n)} = a_0 \exp(in\phi)$, where a_0 is the constant, while ϕ is determined from the following equation:

$$\left(\omega_{0,1}^2 - \omega^2 \right) = -\omega_{0,1}^2 \frac{4}{3\pi} \frac{a^3}{J_1^2(\lambda_1)b^2 d} \times$$

$$\times \left[\rho_0(\omega) + \sum_{k=1}^{\infty} \rho_k(\omega) \cos(k\phi) \right], \tag{32}$$

where

$$\rho_0(\omega) = \Lambda_{-,1}(\omega), \ \rho_k(\omega) = \left(\Lambda_{-,k+1}(\omega) - \Lambda_{+,k}(\omega) \right).$$

From Eq.(32) it follows that in the general case in order to determine the phase shift between cavities it is necessary that couplings of all disks be taken into account. However, as numerical simulations indicate, the contribution of "long range" couplings is small and one can confine oneself to considering field couplings on the finite number of disks. There, since we had used some symmetry relationships ((23)-(24)), it is necessary to observe the strict correlation between the number of terms in the sum over k in Eq.(32) and the number of equations taken into account in ((27)-(30)).

Results of numerical analysis of Eq.(32) are presented below. Tab.1 gives the calculated values[4] of phase shift (in degrees) per cell for the cavity chains with such geometrical dimensions that ensure phase shifts to occur close to $\phi_0 = 2\pi/3$, $\pi/2$, and $\pi/3$. The operation frequency is $f_0 = 2797.0\,\mathrm{MHz}$ ($\lambda_0 = 10.7183\,\mathrm{cm}$). The column (1-0) presents the results of calculations on the base of Eq.(32) at $k = 1$ in the case when fields of various disks do not couple (in Eqs.(27,30) $G_{n,s}^{(2,i)} = 0$ ($i = 1, 2$)). The column (1) presents the results of calculations at $k = 1$ in the case when only "self-coupling" of the first disk fields is taken into account in Eqs.(27,30) (the 1-st disk field couples with the field on the right cross-section of the (-1)-th disk as described by the terms $w_{-,s}^{(1)}G_{n,s}^{(2,2)}$ in (27) and $w_{-,s}^{(1)}G_{n,s}^{(2,1)}$ in (28)). The columns (2)-(4) present the results of calculations at $k = 2, 3, 4$. Considering our transformation of the infinite equation string into the semi-infinite one, we have: the results in column (1-0) correspond to the case of non-coupling disks, (1) — two disks, (2) — four, (3) — six, (4) — eight disks are coupled.

From Tab.(1) it follows that the influence of coupling of different disk fields on the buildup of a certain phase shift depends both on the spacing of the disks and on the hole dimensions. Thus, for instance, in the case of disk-loaded structures operating in the $\phi_0 = \pi/3$ mode, even at small values of the hole radius, it is necessary to take into account field coupling of four disks, while at large one — six disks. In the case of disk-loaded structures operating in the $\phi_0 = \pi/2$ mode it is necessary to take into account field coupling of four disks. In the case of the most commonly used disk-loaded structures operating in the $\phi_0 = 2\pi/3$ mode only coupling of fields of two disks should be taken into account for a broad range of hole radii.

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[4]Our results are in good agreement both with the experimental data, given in [3], and with the calculation results performed within the program developed on the base of partial region technique [17]

Table 1: Calculated values of the phase shift (°) per one cavity for various cavity chains

a/λ_0	(1-0)	(1)	(2)	(3)	(4)
$D = \lambda_0/3$					
0.08	120.1630	120.0249	120.0119	120.0119	120.0119
0.14	120.5835	120.0710	120.0123	120.0126	120.0126
$D = \lambda_0/4$					
0.08	88.6985	90.1831	90.0103	90.0118	90.0118
0.11	88.4270	90.4579	89.9777	89.9859	89.9859
0.14	88.3926	90.9189	89.9929	90.0174	90.0176
$D = \lambda_0/6$					
0.08	54.8735	61.9938	60.0590	60.0650	60.0655
0.11	55.8546	63.8105	60.0845	60.0782	60.0833
0.14	57.4583	65.7765	60.1612	60.0813	60.1005

The dependence of corresponding coefficients on frequency, in general, seriously influences on the electrodynamic characteristics of the system under consideration. Tab.2 presents calculation results of the relationship of phase shift (°) per cavity versus frequency (dispersion relation) for a homogeneous disk-loaded wavequide with $D = \lambda_0/3$ and $a/\lambda_0 = 0.14$. The column (ω) corresponds to the case $\rho_i = \rho_i(\omega)$ ($i = 0, 1$), column ($\omega = \omega_{010}$) — $\rho_i = \rho_i(\omega_{010})$, column ($\omega = 0$) — $\rho = \rho_i(0)$ (quasistatic case). From the calculations it follows that $\rho_i$vs.ω relationship, even within the passband (cf.results in columns (ω) and ($\omega = \omega_{010}$)), exercises an influence upon phase shift.

From the results above, one can deduce that cavity chains of any geometry (homogeneous and inhomogeneous) with $D \geq \lambda_0/3$ and $a/\lambda_0 \leq 0.14$ can be described very accurately by the coupled circuit model, wherein each resonant circuit is coupled to two :

$$\left(\omega_{0,1}^{(i)2} - \omega^2\right) a_{0,1}^{(i)} = -$$

$$- \omega_{0,1}^{(i)2} \left[a_{0,1}^{(i)}\Gamma^{(i)}(\omega) - \left(a_{0,1}^{(i+1)}\Gamma_+^{(i)}(\omega) + a_{0,1}^{(i-1)}\Gamma_-^{(i)}(\omega)\right)\right], \tag{33}$$

where the coefficients $\Gamma^{(i)}, \Gamma_+^{(i)}, \Gamma_-^{(i)}$ for i-th cavity will be determined by two values of the radii of coupling holes, through which this cavity is connected with adjacent ones, geometrical dimensions of the $(i - 1, i, i + 1)$-th cavities and frequency. The results of studies of inhomogeneous cavity chains on the base of Eq.(33) will be presented in a future paper.

5   Conclusion

In this paper on the base of a rigorous electrodynamic approach we have developed a mathematical model of a cylindrical cavity chain with electric coupling. This model combines the model of the equivalent coupled circuit chain and an accurate description of the non-resonant field influence. The above approach can be also used in the case of

Table 2: Phase shift (°) vs frequency for a cavity chain with $D = \lambda_0/3$ $a/\lambda_0 = 0.14$

$f,$	(ω)	$(\omega = \omega_{010})$	$(\omega = 0)$
2.727	25.15	25.23	27.59
2.737	45.12	45.24	48.19
2.747	59.63	59.80	63.50
2.757	72.20	72.48	77.01
2.767	84.01	84.35	89.84
2.777	95.57	96.03	102.72
2.787	107.39	108.01	116.35
2.797	120.07	120.93	131.94
2.807	134.73	136.02	153.34
2.877	155.07	157.81	

magnetic coupling. In this case the problem of accurate description of the potential fields on the holes and slots (see, for example, [18]) will be easier, because within the frame of the partial cross-over regions method the subset of irrotational modes is a part of the complete set of modes that one has to use to expand fields with. This technique is easily transformed for the case of inhomogeneous structures. Then, there is a possibility to control rigorously the effects of "long-range" coupling of cavities.

To this day, the equivalent circuit model was an only approximate one at large couplings. In this case, one had to determine the circuit chain parameters from the measured dispersion curves of the passbands. The above method imbues one with hope that this model can give sufficiently accurate description of the characteristics of the coupled cavity chain at large couplings.

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